Proof in Middle School: Moving Beyond Examples

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Many consider proof to be central to the discipline of mathematics and to the practice of mathematicians, yet surprisingly, the role of proof in school mathematics has traditionally been limited to the domain of high school geometry. This absence of proof in school mathematics has not gone unnoticed and, in fact, has been a target of criticism. Sowder and Harel (1998), for example, argue against limiting students' experiences with proof to geometry: "It seems clear that to delay exposure to reason-giving until the secondary-school geometry course and to expect at that point an instant appreciation for the more sophisticated mathematical justifications is an unreasonable expectation" (674).

Reflecting an awareness of such criticism, as well as embracing the important role proof plays in learning mathematics, recent reform efforts have called for substantial changes in students' experiences with respect to proof. In particular, the *Principles and Standards for School Mathematics* (NCTM 2000) recommends that "Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (56). During middle school, it is expected that students will "examine patterns and structures to detect regularities; formulate generalizations and conjectures about observed regularities; evaluate conjectures; [and] construct and evaluate mathematical arguments" (262). These recommendations, however, pose serious challenges for both students and teachers as the recommended practices represent a significant departure from typical middle school mathematics practices.

The goal of this article is to help teachers both recognize and capitalize on classroom opportunities to meaningfully engage students in proving. We first discuss students' thoughts about proof and their abilities to construct arguments, and then offer suggestions for teachers intended to support their efforts to provide students with meaningful experiences with proof throughout their middle school mathematics education.

What does *Proof* Mean to Middle School Students?

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How do you think your students might respond to the following question: *How do you know when something is right in mathematics?* Take a minute to think about the responses you would expect from your students (as well as the responses you would like to see from your students). Or better yet, think about the responses you would expect and then actually ask your students to respond to the question.

As part of a written assessment about justification and proof, we (authors) posed that very question to approximately four hundred middle school students[1] and although we received a variety of responses, the following are both representative and informative.

> "You experiment until you find five examples to see if the answer is right." (Grade 6 student)
>
> "When you get your test back." (Grade 6 student).
>
> "You don't really know if anything is right, you just have to hope and pray that you are right." (Grade 7 student)
>
> "Because of the Book.[2]" (Grade 7 student)
>
> "If it is called a theory or theorem you know it's true." (Grade 8 student)
>
> "If you are smart you will know." (Grade 8 student)

Although these responses are somewhat humorous (and perhaps not all that surprising), they are also somewhat telling: based on the majority of students' responses (and reflected in the responses above), it seems relatively clear that many students have not

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[1] The students all attended the same middle school in a mid-sized Midwestern city. The demographic breakdown of the school's student population was as follows: 29% African American, 8% Hispanic, 12% Asian, and 51% White.

[2] It is not clear whether this student is referring to a textbook or the "Book," the fictional book the famous mathematician Paul Erdös referred to as the one with all the best and most elegant proofs (i.e., the *Book*!). ☺ See Hoffman (1998) to learn more about Erdös.

given much thought to how one comes to know in mathematics. In fact, five of the six sample responses are not even based on a mathematical rationale, but rather on factors not even under a student's control (e.g., "you know" only when a teacher corrects your test). Such responses are likely due in large part to a lack of ongoing and consistent experiences with reasoning and proof—experiences that help foster an awareness of reasoning as an important means to validate mathematical activity. Overall, students' responses to the question suggest that, to this point in their education, proof has likely not played a meaningful role in their mathematical experiences.

Students' Construction of Mathematical Arguments

So how do students do when asked to construct mathematical arguments? We presented a variety of tasks to the middle school students in which they were asked to justify their response to each task. In this section, we briefly discuss two primary findings and illustrate these findings using one of the tasks presented to students. The findings will then serve as a springboard for our discussion of ways in which teachers can support the development of students' understandings of proof.

The most predominant justification method used by students at all three grade levels was to offer examples as proof that a claim is true. For example, in one task students were asked to respond to the following question and to justify their response: *If you add any three odd numbers together, is your answer always odd?* The following responses are representative of the examples-based justifications students provided:

> "Yes because... 7+7+7=21; 3+3+3=9; 13+13+13=39. Those problems are proof that it is true." (Grade 6 student)

> "1+3+3=7. 3+11+1=15. Yes it would be but you will have to do it a 100 times just to make sure." (Grade 7 student)
>
> "3+5+7=15; 15+13+11=39; 1+3+5=9. Yes, because if you just try to add any odd numbers with each other you'll get an odd answer." (Grade 8 student)

It is interesting that the Grade 7 student only tested two sets of examples, but noted further that one would need to test 100 sets of examples to be sure.[3] In this case the student seems wary of making a conclusive statement based on a small set of examples (a good thing!), however, the student fails to recognize that using a larger set of examples still does not constitute proof that the sum of any three odd numbers is always an odd number.

Although a significant proportion of the students relied on examples as their primary means of justification for the various tasks, there were many students who were able to provide justifications that treated the general case. The following student responses to the task presented above are representative of one such general argument:

> "If you add two odds, the result is even. An even plus one more odd is odd. So three odds added together always result in odd." (Grade 6 student)
>
> "We know that odd and odd equal even. So even (2 odds) added together with odd equals odd. This shows us that no matter what three odd numbers you add together, the sum will always be an odd number." (Grade 7 student)

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[3] See Knuth, Choppin, and Bieda (In press) for a detailed discussion of different types of examples-based arguments.

> "The reason that when you add three odd numbers together you always get an odd, is that odd + odd = even. That gets rid of two odds. The answer is even, plus another odd would always be odd. This is because odd + even = odd." (Grade 8 student)

These arguments are based on the use of two accepted truths (i.e., well known facts for most middle school students)—the sum of two odd numbers is an even number, and the sum of an even number and an odd number is an odd number. In constructing their deductive arguments, these students implicitly employed the associative property to partition the three addends, determined the parity of the partial sum (the first two addends) using the accepted truth that the sum of two odd numbers is even, and then deduced the sign of the final sum using the accepted truth that the sum of an even number (the sum of the first two addends) and an odd number (the third addend) is odd.

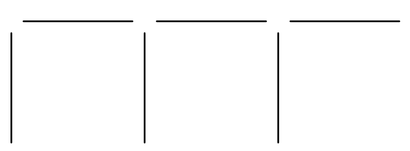
In contrast to the examples-based arguments presented earlier, which prove that the sum of three odd numbers is an odd number only for those cases actually tested, the general argument proves that the sum is an odd number for any three odd numbers. Although more students provided examples-based arguments than provided general arguments on all of the tasks, the fact that students at each grade level were capable of constructing the latter is encouraging. In the following section we discuss suggestions for instruction that may enable more students to move beyond examples as their primary means of justification and toward more general types of arguments (i.e., proofs).

Capitalizing on Classroom Opportunities to Prove

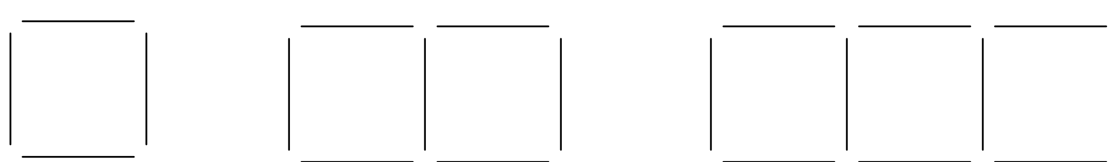
A finding that predominates the results of many studies is students' reliance on the use of examples to prove the truth of a statement or result (e.g., Healy and Hoyles 2000, Porteous 1990). Yet, clearly, our goal for students is to help them learn to move beyond examples, to recognize the limitation of inductive reasoning, and to move toward more mathematically sophisticated means of justifying their claims. So as teachers, how do we help students achieve this goal?

Reasoning based on Structural Characteristics

Consider the following task—a task that is similar to tasks often used in middle school mathematics classrooms. *If a side of each square in the shape below is 1 toothpick, then it takes 10 toothpicks to make this 3 squares-in-a-row shape.*

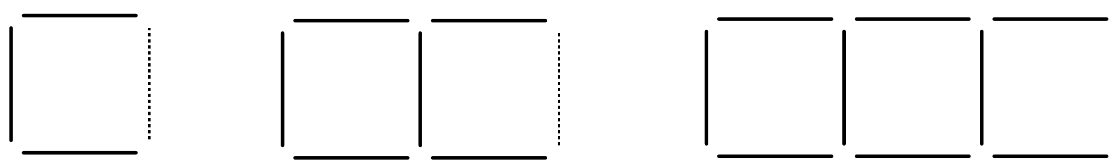


How many toothpicks are needed to build any number of squares-in-a-row? A common approach used by many students is to search for a pattern based on several examples.



Students might first determine the number of toothpicks required for the first few shapes (in this case, the first three shapes), and may recognize that the pattern seems to be three times the number of squares-in-a-row plus one (i.e., 3n + 1). They may then go on to test their generalization with the next shape or two in the pattern. In this example, students often justify their generalization by stating that it worked for all of the cases they tested—an examples-based justification.

Another approach might consider characteristics of the shape itself. For example, we can imagine each square being comprised of a “C-shaped” part consisting of three toothpicks and an end piece of one toothpick (see below).



In the case of 1 square-in-a-row, we have 3+1= 4 toothpicks; in the case of 2 squares-in-a-row, we have 3+3+1=7 toothpicks; and in the case of 3 squares-in-a-row, we have 3+3+3+1=10 toothpicks. Students utilizing this approach are basing their generalization on the mathematical structure of the shape rather than on several examples.[4]

As students share such approaches (the pattern recognition approach and the structural approach) with the class, the opportunity arises to engage them in conversation about the nature of the justifications provided and, in particular, about how the examples-based justification does not guarantee that the generalization holds for all cases whereas the structurally-based justification does guarantee (prove) that the generalization will hold for all cases.

Given the propensity of pattern recognition tasks (e.g., squares-in-a-row task) and number theoretic tasks (e.g., sum of three odd numbers task) in middle school curricula, teachers have ample opportunities to help their students develop more mathematically

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[4] Note that a similar type of structurally-based justification can be generated for the three odd number problem presented earlier (we leave this as an exercise for the reader ☺).

sophisticated means of justifying and proving. And it is likely that such opportunities will not only help students learn to appreciate what constitutes proof in mathematics, but also better prepare them for proof-intensive courses in high school.

Understanding the Role of Examples as Justification

Pattern generalization tasks (such as the squares-in-a-row example) are a staple of middle school mathematics curricula. In the vast majority of cases, the generalizations students determine using examples-based reasoning are correct, thus it is often difficult for students to understand the limitations of such reasoning. One way of helping students to develop an understanding about the limitations of examples-based reasoning is to present them with tasks in which generalizing from several cases actually does not lead to a correct generalization (see, for example, the illustration presented on pages 266-267 of the *Principles and Standards for School Mathematics*).

It is also important to discuss with students the role examples play in proving propositions or statements. For example, consider the following problem from a 6th grade lesson in the *Connected Mathematics* curriculum:

> Which of the following statements are *always true*, which are *never true*, and which are *sometimes true*. Explain your reasoning.
>
> > a) If a number is greater than a second number, then the first number has more factors than the second number.
> >
> > b) The sum of two odd numbers is even.

This problem provides an excellent opportunity for a teacher and students to discuss the contrasting situations: in one case, an example is enough to prove a statement is false, and in the other case, examples are not enough to prove a statement is true. In the former, one need only consider a single pair of numbers, such as 6 and 7, to prove that the statement is false. In the latter, the use of examples is never enough to prove that the statement is true.[5]

Presenting students with a variety of tasks in which examples play different roles can serve to help students develop an appreciation and understanding for their use as a means of justification. Tasks such as the one presented in the *Principles and Standards for School Mathematics* can help students develop "a healthy skepticism in their work with patterns and generalization" (NCTM 2000, 267), that is, to recognize that examples are not sufficient to prove the truth of a generalization (although they do play an important role in making conjectures). Likewise, tasks such as the one presented in the *Connected Mathematics* curriculum can help students develop an appreciation for the role of examples in (dis)proving a claim.

Concluding Remarks

As van Dormolen (1977) aptly stated, "not until we manage to teach our students what giving a proof really means can we expect them to give deductive arguments with understanding and insight" (33). An important part of teaching students what giving a proof really means involves providing opportunities for them to engage in discussions about the nature of different justifications (e.g., examples-based justifications,

[5] It is also important to note that there are situations in which examples can prove the truth of a mathematical statement (i.e., proof by exhaustion). Consider, for example, the following statement: There are three prime numbers between 5 and 15. In this case, students could simply check each number and show that the only prime numbers are 7, 11, and 13, thus proving the truth of the statement by checking (all of the possible) examples.

structurally-based justifications), helping them to understand the role of examples in proving, and creating a classroom culture in which reasoning, explaining, and justifying are a regular and consistent part of doing mathematics.

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